8.2 Orthogonal Polynomials and Least Squares Approximations

Parallel with Finite-Dimensional vector Spaces

Consider the two $\mathbb{R}^3$ bases

$$B_1 = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix} \right\}$$

$$B_2 = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$$

a) Coordinates of $\begin{bmatrix}5\\4\\-10\end{bmatrix}$ in B_1: $c_1\begin{bmatrix}1\\1\\1\end{bmatrix}+c_2\begin{bmatrix}1\\0\\1\end{bmatrix}+c_3\begin{bmatrix}1\\2\\3\end{bmatrix}= \begin{bmatrix}5\\4\\-10\end{bmatrix}$

$$\begin{bmatrix}1&1&1\\1&0&2\\1&1&3\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}5\\4\\-10\end{bmatrix}$$

We need to Solve linear System:

$$\begin{cases} c_1 + c_2 + c_3 = 5 \\ c_1 \qquad + 2c_3 = 4 \\ c_1 + c_2 + 3c_3 = -10 \end{cases}$$

b) Coordinates of $\begin{bmatrix} 5 \\ 4 \\ -10 \end{bmatrix}$ in B_2

B_2 is an orthogonal basis

In fact,

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = -1+1=0$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0, \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

Then, to find the coords: d_1, d_2, d_3

or
$$d_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + d_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -10 \end{bmatrix} \quad (2.1)$$

We just need to dot product the basis vectors with (2.1) and solve for the d_i's.

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \left(d_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + d_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \\ -10 \end{bmatrix}$$

$$= d_1 (2) + d_2 (0) + d_3 (0) = 5+4=9$$

$$\Rightarrow \boxed{d_1 = 9/2}$$

Similarly, $\boxed{d_2 = -1/2}$, $\boxed{d_3 = -10}$

Approximation of $f \in C[a,b]$ by polynomials degree at most n.

Let $$P_n(x) = \sum_{k=0}^{n} a_k x^k$$

We define the least squares error by approximating $f(x)$ by $P_n(x)$ as

$$\boxed{E(a_0, a_1, .., a_n) = \int_a^b \left[f(x) - \sum_{k=0}^{n} a_k x^k \right]^2 dx} \quad (3.1)$$

Natural extension of discrete case.

$$\sum_{i=1}^{m} \longrightarrow \int_a^b$$

We want to minimize $E(a_0, .., a_n)$

Necessary condition:

$$\frac{\partial E}{\partial a_j} = 0, \quad j = 0, 1, .., n.$$

Now,

$$0 = \frac{\partial E}{\partial a_j} = 2 \int_a^b \left[f(x) - \sum_{k=0}^{n} a_k x^k \right] (-x_j)\, dx$$

which is equivalent to

$$\boxed{\sum_{k=0}^{n} a_k \int_a^b x^{k+j} dx = \int_a^b f(x) x^j dx \qquad j = 0, 1, .., n} \quad (3.2)$$

Example 1. Approximate $f(x) = \sin(\pi x)$ in $[0,1]$ by a 2^{nd} degree polynomial: $\boxed{P_2(x) = a_2 x^2 + a_1 x + a_0.}$

Using the normal equations (3.2)

$$j=0: \quad a_0 \int_0^1 1\,dx + a_1 \int_0^1 x\,dx + a_2 \int_0^1 x^2 dx = \int_0^1 \sin(\pi x)\,dx$$

$$j=1: \quad a_0 \int_0^1 x\,dx + a_1 \int_0^1 x^2 dx + a_2 \int_0^1 x^3 dx = \int_0^1 x \sin(\pi x)\,dx$$

$$j=2: \quad a_0 \int_0^1 x^2 dx + a_1 \int_0^1 x^3 dx + a_2 \int_0^1 x^4 dx = \int_0^1 x^2 \sin(\pi x)\,dx$$

The equations after integration:

$$\begin{cases} a_0 + \dfrac{a_1}{2} + \dfrac{a_2}{3} = \dfrac{2}{\pi} \\[2ex] \dfrac{1}{2} a_0 + \dfrac{1}{3} a_1 + \dfrac{1}{4} a_2 = 1/\pi \\[2ex] \dfrac{1}{3} a_0 + \dfrac{1}{4} a_1 + \dfrac{1}{5} a_2 = \dfrac{\pi^2 - 4}{\pi^3} \end{cases}$$

After solving this system

$$\boxed{P_2(x) = -4.12..\,x^2 + 4.12..\,x - 0.05..}$$

Disadvantages:

1) Matrix of linear system obtained from (3.2) is know as Hilbert matrix is ill-conditioned (round-off error).

2) Increasing the degree of the approximating polynomial requires solving a larger system of equations. It does not use the lower order polynomials.

Definition: The set $\Pi_n[a,b]$ all polynomials of degree at most n in $[a,b]$.

Theorem: The set $\Pi_n[a,b]$ is a vector space under the sum of functions and product by scalars.

Clearly, the subset: $B = \{1, x, x^2, \dots, x^n\}$ of $\Pi_n[a,b]$ is a basis for Π_n. Since it is lin. indep and spans Π_n.

Another Convenient basis for $\Pi_3^{[-1,1]}$ is given by

$$B_2 = \left\{ \begin{array}{l} \phi_0(x) = 1,\ \phi_1(x) = x,\ \phi_2(x) = x^2 - 1/3 \\ \qquad\qquad \phi_3(x) = x^3 - \frac{3}{5}x \end{array} \right\}$$

The important property of B_2 is that it is <u>an orthogonal basis</u>.

<u>Definition</u>: A set of functions $S = \{\phi_1, , \phi_n\}$ is said to be orthogonal with respect to a weight function $w(x)$ if

$$\int_a^b w(x)\,\phi_j(x)\,\phi_k(x)\,dx = \begin{cases} 0, & j \neq k \\ \alpha_k > 0, & j = k. \end{cases}$$

If $\alpha_k = 1$. S is said to be an orthonormal basis.

$w(x)$ is integrable in $[a,b]$, and $w(x) \geq 0$ for all x, and $w(x) \not\equiv 0$ for any subinterval of $[a,b]$.

The basis B_2 of Legendre polynomials is an orthogonal basis. In fact,

Inner product

$$\int_{-1}^{1} \phi_1(x)\,\phi_2(x)\,dx = \int_{-1}^{1} x\left(x^2 - 1/3\right)dx$$

$$= \int_{-1}^{1}\left(x^3 - \frac{1}{3}x\right)dx = \frac{x^4}{4} - \frac{1}{3}\frac{x^2}{2}\Big|_{-1}^{1} = 0$$

Also, $\displaystyle\int_{-1}^{1} \phi_0(x)\,\phi_1(x)\,dx = \int_{-1}^{1} x\,dx = 0.$

$$\int_{-1}^{1} \phi_0(x)\,\phi_2(x)\,dx = 0, \quad \int_{-1}^{1} \phi_0(x)\,\phi_3(x)\,dx = 0, \ldots$$

Least squares Approximations by means of Orthogonal fns.

There are two main advantages using orthog. bases

i) There is no need to solve a large linear system.

ii) The approximation by higher order degree polyn. can be obtained from the lower ones by just adding one more term.

Definition:

$$\boxed{E(a_0,\ldots,a_n)=\int_a^b w(x)\left[f(x)-\sum_{k=0}^{n} a_k\phi_k(x)\right]^2 dx}$$

Least squares error by approximating $f(x)$ in $[a,b]$ by a linear combination of the functions $\phi_k(x)$. $i=0,\ldots,n$.

Minimization

Necessary Condition: $\boxed{\frac{\partial E}{\partial a_j}=0,\quad j=0,\ldots,n}$

Which is equivalent to

$$2\int_a^b w(x)\left[f(x)-\sum_{k=0}^{n} a_k\phi_k(x)\right](-\phi_j(x))\,dx = 0.$$

or
$$\sum_{k=0}^{n} a_k\int_a^b \phi_k(x)\phi_j(x)w(x)\,dx=\int_a^b w(x)f(x)\phi_j(x)\,dx$$

Using the orthogonality of ϕ_k's.

$$\boxed{a_j\overbrace{\int_a^b \phi_j^2(x)\,w(x)\,dx}^{=\alpha_j}=\int_a^b w(x)f(x)\phi_j(x)\,dx.}$$

Therefore,

$$\boxed{a_j = \frac{1}{\alpha_j} \int_a^b w(x)\, f(x)\, \phi_j(x)\, dx}$$

we don't have to solve a system to compute the a_j's.

Exercise: Approx. $f(x) = \sin(\pi x)$, $x \in [-1,1]$

Using Legendre Polynomials up to degree 3.

$\phi_0(x) = 1$, $\phi_1(x) = x$, $\phi_2(x) = x^2 - 1/3$, $\phi_3(x) = x^3 - \frac{3}{5}x$.

$\sin(\pi x) \simeq a_0 \phi_0(x) + \cdots + a_3 \phi_3(x)$.

Now,
$$a_i = \frac{\int_{-1}^{1} \sin(\pi x)\, \phi_i(x)\, dx}{\int_{-1}^{1} \phi_i^2(x)\, dx}, \quad i = 0, .., 3.$$

Then
$$a_0 = \frac{\int_{-1}^{1} \sin(\pi x)\, dx}{\int_{-1}^{1} 1^2\, dx} = 0$$

$$a_1 = \frac{\int_{-1}^{1} x \sin(\pi x)\, dx}{\int_{-1}^{1} x^2\, dx} = \frac{2/\pi}{2/3} = \frac{3}{\pi} \simeq 0.955$$

$$a_2 = \frac{\int_{-1}^{1} \overbrace{(x^2 - 1/3)\sin(\pi x)}^{\text{odd fn.}}\, dx}{\int_{-1}^{1} (x^2 - 1/3)^2\, dx \neq 0} = \frac{0}{\int_{-1}^{1} (\ )^2 dx} = 0$$

$$a_3 = \frac{\int_{-1}^{1} (x^3 - \frac{3}{5}x) \sin(\pi x)\, dx}{\int (x^3 - \frac{3x}{5})^2\, dx} = \frac{\frac{4}{5} + \frac{\pi^2 - 15}{\pi^3}}{8/175} \simeq -2.8956$$

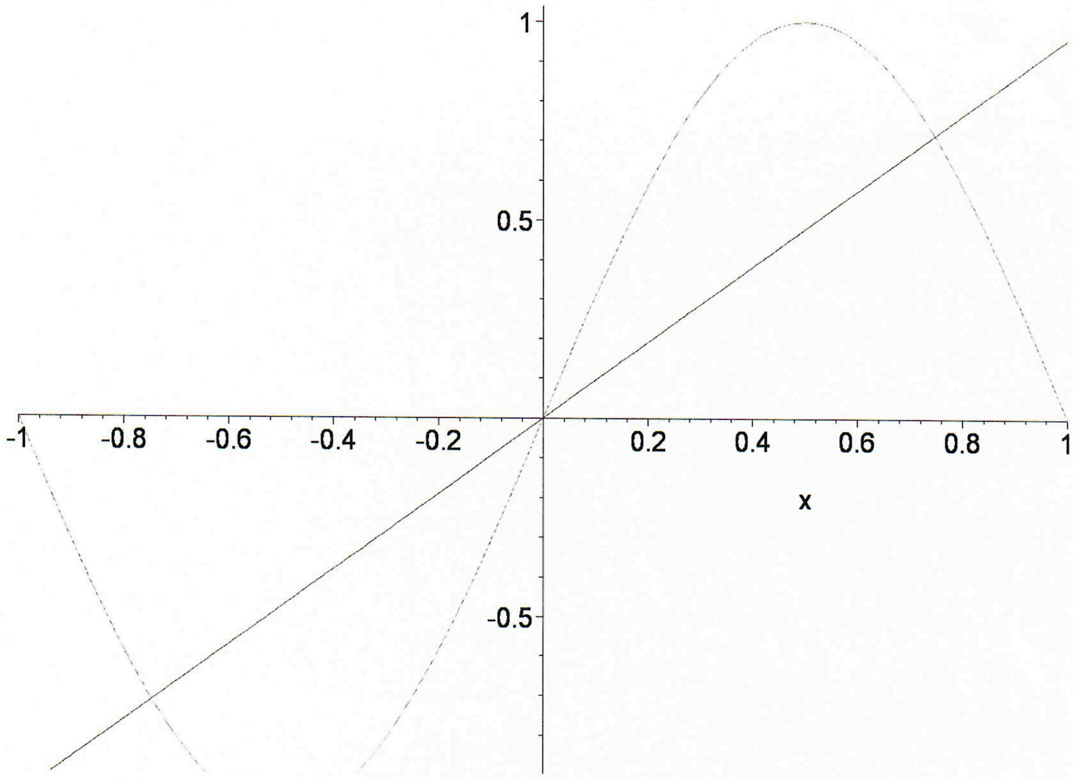
```
> evalf(int(x*sin(Pi*x),x=-1..1));
```

$$.6366197722$$

```
> evalf(int((x^2-1/3)*sin(Pi*x),x=-1..1));
```

$$0.$$

```
> plot({sin(Pi*x),(3/Pi)*x},x=-1..1);
```




```
> int(x^2*sin(pi*x),x);
```

$$\frac{-\pi^2 x^2 \cos(\pi x) + 2\cos(\pi x) + 2\pi x \sin(\pi x)}{\pi^3}$$

```
> int((x^3-(3/5)*x)^2,x=-1..1);
```

$$\frac{8}{175}$$

```
> alpha3:=2/7-12/25+18/75;
```

$$\alpha 3 := \frac{8}{175}$$

```
> I3:=int((x^3-(3/5)*x)*sin(Pi*x),x=-1..1);
```

$$I3 := \frac{4}{5}\,\frac{-15+\pi^2}{\pi^3}$$

```
> a3:=evalf(I3/alpha3);
```

$$a3 := -2.895604778$$

The new least squares approx. of degree three is given by

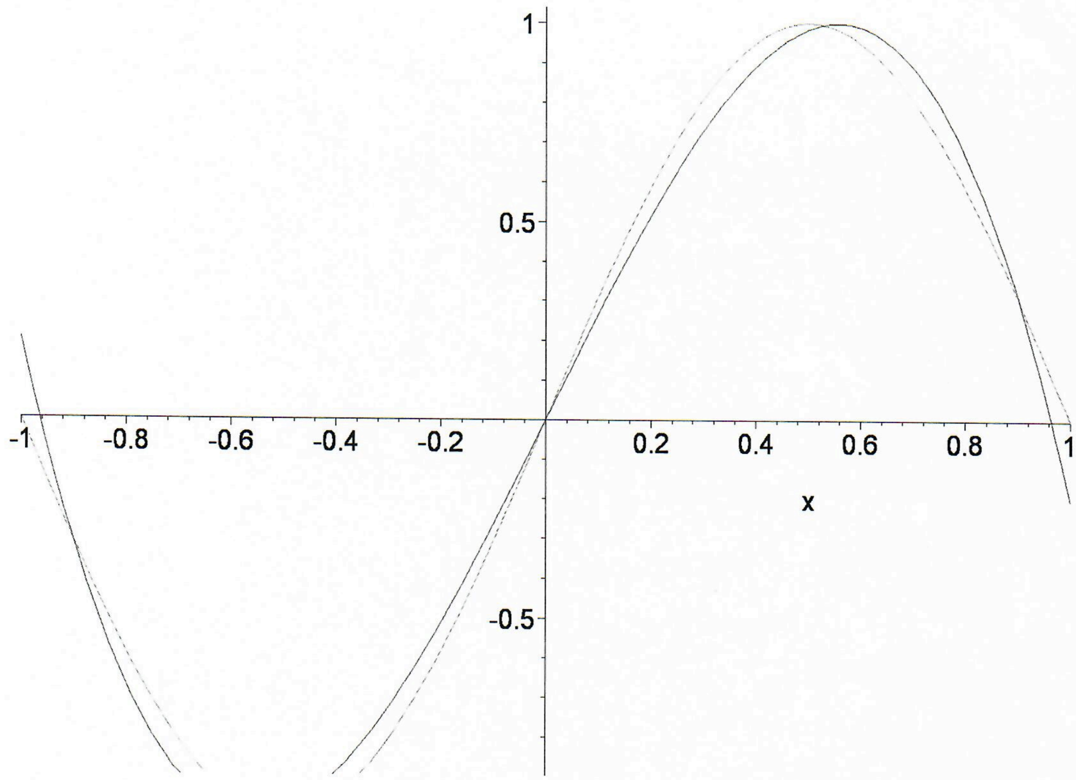
```
> P3:=x->3/Pi*x+a3*(x^3-3/5*x);
```

$$P3 := x \to 3\,\frac{x}{\pi} + a3\left(x^3 - \frac{3}{5}\,x\right)$$

```
> evalf(P3(2));
```

$$-17.78025317$$

```
> plot({sin(Pi*x),P3(x)}, x=-1..1);
```

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Therefore,

$$\sin(\pi x) \simeq 0.955\,x - 2.8956\left(x^3 - \tfrac{3}{5}x\right).$$

If we increase to polynomials degree 5.

$$a_4 = \frac{\int_{-1}^{1} \overbrace{\left(x^4 - 6/7\,x^2 + \frac{3}{35}\right)}^{\text{even fn.}} \sin(\pi x)\,dx}{\int_{-1}^{1}\left(x^4 - 6/7\,x^2 + \frac{3}{35}\right)^2 dx} = 0$$

$$a_5 = \frac{\int_{-1}^{1}\left(x^5 - \frac{10}{9}x^3 + \frac{5}{21}x\right)\sin(\pi x)\,dx}{\int_{-1}^{1}\left(x^5 - \frac{10}{9}x^3 + \frac{5}{21}x\right)^2 dx.} \neq 0.$$